On a momentum-mass flux diagram for turbulent jets, plumes and wakes

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Many salient features of fully developed turbulent jets, plumes and wakes with steady mean flow are shown clearly by the relationship between the momentum flux and the mass flux in the column of moving fluid. Using a simple model for the flows, this relationship can be found from the solution of a single ordinary differential equation and the character of many related flows can be represented immediately (except for actual distribution in space) on a single momentum-mass flux diagram.

In this note some approximate solutions based on dimensional arguments are outlined briefly for cases of buoyant and non-buoyant wakes and jets directed along the axis of a uniform main stream, and momentum-mass flux curves are presented.

1. Introduction

There are sufficiently strong similarities in the behaviour of round turbulent jets and wakes in a still fluid environment or in a steady ambient stream to suggest that, for at least some purposes, this family of columnar turbulent flows can usefully be investigated as a single group. Further members of the same family are provided by buoyant wakes and plumes (i.e. buoyant jets) directed along the line of the (uniform) force field which produces the buoyancy. For each of these columnar turbulent flows a detailed treatment can be given using an approximate method described by Morton, Taylor & Turner (1956), but sufficient information for many purposes can be obtained from the same assumptions more simply by relating the momentum flow and the mass flow in the column and eliminating, in so far as this is possible, other variables. In particular, this partial solution is useful either for a comparison of different columnar flows, or for finding *whether* flows exhibit specified types of behaviour rather than *where* they may do so.

There are, of course, differences between the various flows of the family which must be borne in mind. Each flow will take a certain distance from the source to develop fully its appropriate turbulent character, and the flow in this initial region will be partly characteristic of the source and may not be fully turbulent (cf. Kuethe 1935, for the jet, and Goldstein 1938, chap. XIII, for the wake). The flow in this 'entry' or 'development' region is quite outside the scope of a treatment based on developed turbulent flow, but this is of no great importance if in each case the flow is conceived as originating from an appropriate virtual source, the position of which is determined by the fluxes of mass, momentum, etc. at the end of the development region. In some cases source characteristics may persist for much greater distances downstream, as with the oscillations of a wake, which are specially marked over certain ranges of Reynolds number; these effects are more difficult to make allowance for, but this should to some extent be possible by suitable modification of the virtual source strength. Another difference arises from the fact that with increasing distance from the virtual source the local Reynolds number remains constant for a simple jet, increases for a simple plume, and decreases for a simple wake; hence the level of turbulence will decay far from the source of a wake.

The common features of jets, plumes and wakes make possible a single type of treatment which can be based on a common set of assumptions. All these flows show a relatively slow rate of spread with increasing distance from the source, and this suggests that mean profiles across the column of longitudinal velocity, etc., will be similar in shape at all stations along the column (such profiles are known experimentally to be approximately Gaussian). This assumption is certainly reasonable in the cases of simple jets and simple wakes; it is at least plausible in cases where there is an outer flow and may be used until further information on these flows is available. It will also be reasonable to adopt the usual boundary layer assumption that longitudinal diffusion is negligible relative to lateral diffusion; and molecular diffusion will in all cases be neglected relative to turbulent diffusion. The flow will be taken as independent of Reynolds number, and also of the relative density difference parameter (characteristic density difference divided by reference density) except in so far as density variations give rise to buoyancy forces. The turbulent mixing or entrainment of ambient fluid can now be fully represented by a speed of flow into the column at some arbitrarily defined 'mean outer edge', and as a consequence of the assumptions already made this inflow speed across the 'mean boundary' of the column must be proportional to the magnitude of the difference between a characteristic mean velocity along the column and the (parallel) velocity of the ambient fluid at a distance. This basic assumption follows immediately from dimensional considerations and must be valid wherever the physical character of the flow is adequately represented by the previous assumptions. It bears no immediate relationship to mixing length theories other than its dimensional background, and it is in fact a good deal weaker since no interrelationship is imposed on mixing processes at different parts of a section of the column. The result of assuming the inflow rate and the similarity of profiles is to suppress all detail of the *transverse* structure of the column from the solution; hence *any* profile shape can be used without loss of additional physical information, and although profiles are known to be almost Gaussian it will be simplest here to work in terms of *top-hat* profiles in which variables take a constant value (different to that in the ambient stream) across the whole width of the column.

2. Formulation

A convenient frame of reference for treatment of the axially symmetric flows that will be considered here has origin at the virtual source (as yet undetermined) and x-axis directed along the axis of the column. Take $r = r(x)$ as the mean column radius, $u = u(x)$ as the mean velocity along the column (which for the top-hat profile is uniform over column sections), u_0 as the main stream velocity of the ambient fluid assumed to be parallel to the column axis, $\rho = \rho(x)$ as the density of column fluid and ρ_0 the ambient density. Although u_0 and ρ_0 can be permitted to vary with x, they will be regarded as constant here for simplicity. Then, if the buoyancy force arises from the action of a uniform gravitational field acting in the negative x-direction, equations representing the conservation of mass, momentum, and density deficiency (e.g. due to transported heat) can be written as

$$\left.\begin{aligned} \frac{d}{dx}(\pi\rho r^2 u) &= 2\pi\rho r E|u-u_0|, \\ \frac{d}{dx}(\pi\rho r^2 u^2) &= 2\pi\rho r E|u-u_0|\,u_0 + \pi g r^2(\rho_0-\rho), \\ \frac{d}{dx}\{\pi r^2 u g(\rho_0-\rho)\} &= 0, \end{aligned}\right\} \quad (1)$$

where E is the entrainment constant, which is defined as the ratio of the inflow speed at distance r to the speed difference between column and ambient flows, calculated in terms of the top-hat profile. It may be observed that these equations have been derived on the assumption that all entrainment is due to turbulent mixing outwards from within the column into the laminar ambient flow. Fluid flowing in the main stream cannot of itself enter the column, but will merely carry the column boundary before it until engulfed by the column turbulence.

Equations (1) can be written more concisely in terms of the variables

$$v = r^2 u, \quad m = r^2 u^2, \quad b = r^2 u g(\rho_0 - \rho)/\rho_0,$$

which are porportional to the fluxes of mass, momentum and buoyancy, respectively. The further set of transformations to the non-dimensional variables V, M, X, given by

$$v = v_0 V, \quad m = m_0 M, \quad x = (v_0/2Em_0^{\frac{1}{2}})X,$$

is based on the measurable values v_0 for v and m_0 for m at the end of the development region, that is at the first point of the column which can be regarded as characteristic of the developed turbulent flow, say at the point $x = x_0$. Under these two sets of transformations, equations (1) reduce to

$$\left.\begin{aligned} \frac{dV}{dx} &= \frac{V}{M^{\frac{1}{2}}}\left|\frac{M}{V}-A\right|, \\ \frac{dM}{dx} &= \frac{AV}{M^{\frac{1}{2}}}\left|\frac{M}{V}-A\right| + B\frac{V}{M}, \end{aligned}\right\} \quad (2)$$

and $b = b_0 =$ constant, where $A = u_0 v_0/m_0$ is the ratio of the free stream velocity to the mean velocity in the column at $x = x_0$, and

$$B = (b_0 v_0^2)/(2Em_0^{\frac{5}{2}}) = [rg(\rho_0-\rho)/(2E\rho_0 u^2)]_{x=x_0};$$

A is necessarily positive or zero, but B can also be negative. In deriving these equations it has been assumed that variations in density need be taken into account only where they give rise to buoyancy forces. A solution to equations (2) subject to the conditions $V = 1$, $M = 1$ at the appropriate point (though it is not obvious where this is) could be found without difficulty for any particular case (A, B), but it is very much simpler to carry out a survey of solution types if X is first eliminated to give the single equation

$$\frac{dM}{dV} = A + \frac{BV}{M^{\frac{1}{2}}|M - AV|}, \tag{3}$$

subject to the condition $M = 1$ when $V = 1$. Solutions to equations (3) will demonstrate the character of appropriate columnar flows without giving their distribution in space, so that it is no longer necessary to known *where* the boundary condition is to be applied. Once a solution to (3) has been found, the corresponding distribution in space is obtained by integrating the equation

$$\frac{dX}{dV} = \frac{M^{\frac{1}{2}}}{|M - AV|}, \tag{4}$$

subject to the condition that $X = 0$ when V takes the value appropriate to the virtual source (and this value is given by the solution to equation 3). The whole solution is then determined in the parametric form $X = X(V)$, $M = M(V)$.

For a survey of the family of columnar flows it will be convenient to define two groups, using slightly more general definitions for jets and wakes than are usual. Columnar flows having $A < 1$ will be termed jets, and those with $A > 1$ will be wakes; this is a classification according to the relative motion of the column and its environment at the beginning of the developed flow, and in jets the column velocity is greater than the ambient velocity while in wakes it is smaller.

In general there will be different types of virtual source for the two groups. This can be seen most readily by considering curves on the momentum–mass flux diagram representing different columnar flows. For each flow there is an M–V curve which passes through the 'initial point,' (1, 1) corresponding to the start of the developed flow. The volume flux V must necessarily increase monotonically with increasing distance along the column from the virtual source since there is no possibility of 'detrainment' (i.e. removal of 'marked' column fluid due to the motion of the ambient fluid, possible only when the ambient flow is turbulent), and indeed it is this monotone increasing character of V with X which makes V a suitable independent parameter for the solution. Thus M is a single-valued function of V, and it follows that each M–V curve must cut either the V-axis in the range $0 < V < 1$ or the M-axis with $0 \leqslant M$ (a negative value of M would correspond to a reversed flow). These points of intersection with the axes represent the virtual sources; intercepts $(0, M)$ on the M-axis correspond in general to virtual sources for jet flows, and intercepts $(V, 0)$ on the V-axis are virtual sources of wakes, though there are exceptions caused by buoyancy effects. It is an advantage of this approach that the strength of the virtual source is given as a natural product of the solution.

The two groups will now be considered in greater detail.

2.1. *Jets*

This group will include all columnar flows for which $u > u_0$ at $x = x_0$, the start of the developed turbulent flow, irrespective of whether or not the initial flow in the column is buoyant. The virtual source for a jet has strength $(0, M, B)$, where B provides a measure of the constant buoyancy flux along the jet. It follows that the virtual source must be a *point* source, since $V \propto r^2 u \propto M/u$, and hence as $x \to 0$, $u \to \infty$ and $r \to 0$.

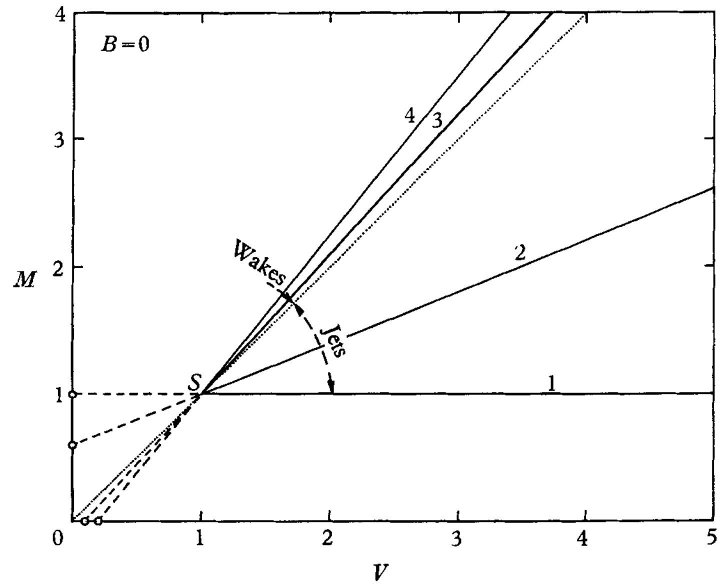

FIGURE 1. A diagram showing in non-dimensional form the relationship between the momentum flux M and the volume flux V for axially symmetrical turbulent jets and wakes released along the flow of a uniform main stream. The point $S(1, 1)$ represents conditions at the equivalent source of the flow, the unbroken lines proceeding to the right of S represent flow in the columns at increasing distance from the source and the broken lines to the left of S cut the axes in points (marked by circles) corresponding with the appropriate virtual sources and represent the hypothetical part of the column flow. The dotted line ($M = V$) provides a division into broad classes of 'jet-like' and 'wake-like' flows. The lines shown represent: (1) the simple jet ($A = 0$) in a still environment; (2) a jet in a uniform stream ($A = 0{\cdot}4$); (3) a forced wake ($A = 1{\cdot}1$) produced by emitting fluid at a mean velocity below that of the main stream; (4) the simple wake ($A = 1{\cdot}25$) of a bluff body. There are no buoyancy effects.

2.1 (*a*). *The simple jet* without buoyancy in a uniform environment at rest has $A = 0$, $B = 0$ and equation (3) reduces to $dM/dV = 0$, with solution $M = 1$, which is merely an expression of the well-known constancy of momentum in the simple jet. The *actual* flow in a simple jet is represented on the $M-V$ diagram by the half-line $M = 1$, $V \geqslant 1$; the virtual source strength is given from the point of intersection of the line $M = 1$ with the M-axis as $(0, 1)$, and the segment of the line for $0 \leqslant V < 1$ represents the *hypothetical* flow between the virtual source and the supposed starting point for the well-developed flow. This behaviour is represented on figure 1, where the actual flow is shown with a continuous line and the hypothetical part of the flow by a broken line. Equation (4) has solution $X = V$; hence the full solution for the simple jet is $V = X$, $M = 1$.

2.1 (*b*). *A jet in a uniform stream* with the jet emitted in the direction of the main stream. In this case $0 < A < 1$ and $B = 0$, and equation (3) has the form $dM/dV = A$ with $M = 1$ at $V = 1$. Hence $M-1 = A(V-1)$, that is, jet momentum increases linearly with increasing volume flow because of the momentum of the fluid entrained. Actual flow in the jet is represented by the half-line of slope A through the point (1, 1) and with $V \geqslant 1$, and the strength of the (virtual) point source is $(0,\ 1-A)$ obtained from the intersection of the line with the M-axis. Possible flows of this kind are given for $0 < A < 1$, and a typical example is illustrated in figure 1 (with a developed jet velocity two and a half times the main stream velocity i.e. $A = \frac{2}{5}$). The solution to equation (4) with $X = 0$ at $V = 0$ is

$$X = \tfrac{2}{3}A^{-1}(1-A)^{-1}\{[1+A(V-1)]^{\frac{3}{2}} - (1-A)^{\frac{3}{2}}\};$$

hence the actual (effective) source is at $X_0 = \frac{2}{3}A^{-1}(1-A)^{-1}\{1-(1-A)^{\frac{3}{2}}\}$. The full solution is

$$u = u_0\frac{(X+L)^{\frac{2}{3}}}{(X+L)^{\frac{2}{3}}-L^{\frac{2}{3}}}, \quad r = \frac{[(1-A)\,m_0]^{\frac{1}{2}}}{u_0}\,\frac{(X+L)^{\frac{2}{3}}-L^{\frac{2}{3}}}{L^{\frac{1}{3}}(X+L)^{\frac{1}{3}}},$$

where $L = \frac{2}{3}(1-A)^{\frac{1}{2}}/A$ and $A = u_0 v_0/m_0$. It may be noted that the asymptotic behaviour of a jet in a uniform stream is given by

$$\frac{u-u_0}{u_0} \sim \frac{L^{\frac{2}{3}}}{(X+L)^{\frac{2}{3}}}\left\{1+\frac{L^{\frac{2}{3}}}{(X+L)^{\frac{2}{3}}}+\ldots\right\},$$

$$\frac{u_0 r}{[(1-A)\,m_0]^{\frac{1}{2}}} \sim \frac{(X+L)^{\frac{1}{3}}}{L^{\frac{1}{3}}}\left\{1-\frac{L^{\frac{2}{3}}}{(X+L)^{\frac{2}{3}}}+\ldots\right\},$$

so that the asymptotic behaviour is that of the wake (see below). Figure 2 shows curves for the dimensionless velocity excess $(u-u_0)/u_0$ in the jet relative to the ambient fluid and for the dimensionless jet radius $u_0 r/[(1-A)\,m_0]^{\frac{1}{2}}$ plotted against dimensionless distance X/L from the virtual source; in each case curves for the simple jet and for the asymptotic wake behaviour are superimposed in the best form for comparison.

2.1 (*c*). *A forced plume* (*or buoyant jet*) in a uniform still environment: the flux of buoyancy is constant and equal to the flux from the source; $A = 0$ and B may take any value, hence equation (3) reduces to

$$\frac{dM}{dV} = \frac{BV}{M^{\frac{3}{2}}},$$

and the solution with $M = 1$ at $V = 1$ is

$$M^{\frac{5}{2}} = 1+\tfrac{5}{4}B(V^2-1).$$

This curve represents for $V > 1$ the flow that might be observed in a buoyant jet; for large values of V, $M \sim (5B/4)^{\frac{2}{5}}V^{\frac{4}{5}}$, and hence for sufficiently large V these curves will always lie under the line $M = V$, indicating jet-like flow. It may be seen by following the curve back ($V < 1$) that there is a normal jet-type point source of strength $(V, M, B) = [0, (1-5B/4)^{\frac{2}{5}},\ B]$ provided that $B < 4/5$, and this includes the case $B < 0$ of negative buoyancy forces; $dM/dV = 0$ at $V = 0$ and the flow in a sufficiently close neighbourhood of the virtual source approximates to that of a simple jet (i.e. is momentum controlled). If, on the other hand, $B > 4/5$ then the curve cuts the V-axis first, and there is a virtual source of strength $[\{1-4/(5B)\}^{\frac{1}{2}}, 0, B]$. This is a virtual source of quite a dif-

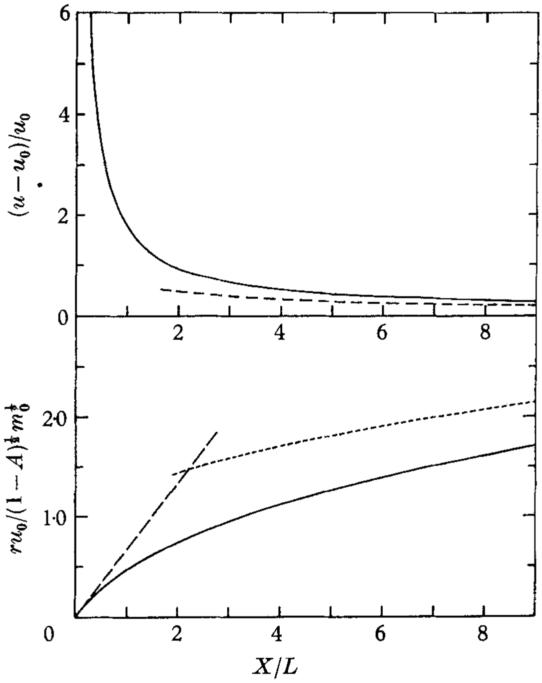

FIGURE 2. The non-dimensional velocity $(u-u_0)/u_0$ and radius $u_0(1-A)^{-\frac{1}{2}}\, m_0^{-\frac{1}{2}}r$ of a jet released along the flow of a uniform stream of speed u_0, plotted against the non-dimensional distance X/L from the virtual source. The broken curves for larger values of X/L represent the asymptotic behaviour of a simple wake, and the broken line for small X/L shows the spread of a simple jet in a still environment.

ferent kind, characteristic of *wake* rather than jet flow; it has infinite radius and zero efflux velocity (see below, under wakes), dM/dV is now infinite at $M = 0$ and the plume is buoyancy controlled in a neighbourhood of the source. A treatment for the forced plume has already been described in detail (Morton 1959*a*). These solutions are illustrated in figure 3 by the curves: (i) $B = -0{\cdot}462$, in which case heavy fluid is discharged and will be continuously decelerated by buoyancy forces (in addition to entrainment effects) until a greatest height is reached ($M = 0$)—the solution cannot be continued beyond this point; (ii) $B = 0{\cdot}185$, a normal example of a jet with positive buoyancy; (iii) $B = 0{\cdot}8$, the case of a simple plume from a source of pure buoyancy, for here $M = V^{\frac{4}{5}}$ and there is a point source $(0, 0, B)$ at $X = 0$; (iv) $B = 1{\cdot}6$, when the virtual source is of 'wake' type and the curve representing the early part of the actual flow lies above the dividing line $M = V$, but falls below it for larger values of V (i.e. reverts to 'jet' behaviour).

2.1 (*d*). *Simple plume in a stably stratified environment* at rest: useful information can be carried on an M–V diagram even in more complicated cases where there are other dependent variables (e.g. buoyancy flux); to illustrate this a curve has been drawn in figure 3 for the simple plume projected upwards into stably stratified ambient fluid. The results (taken from Morton *et al.* 1956) are plotted for $V > 1$ only as they are hard to distinguish from those of the simple plume in $0 \leqslant V \leqslant 1$. There is again a maximum height to which plume fluid can be projected.

2.1 (*e*). *A buoyant jet projected along a uniform upward stream* of uniform density: neither A nor B vanishes, and hence the full solution to equation (3)

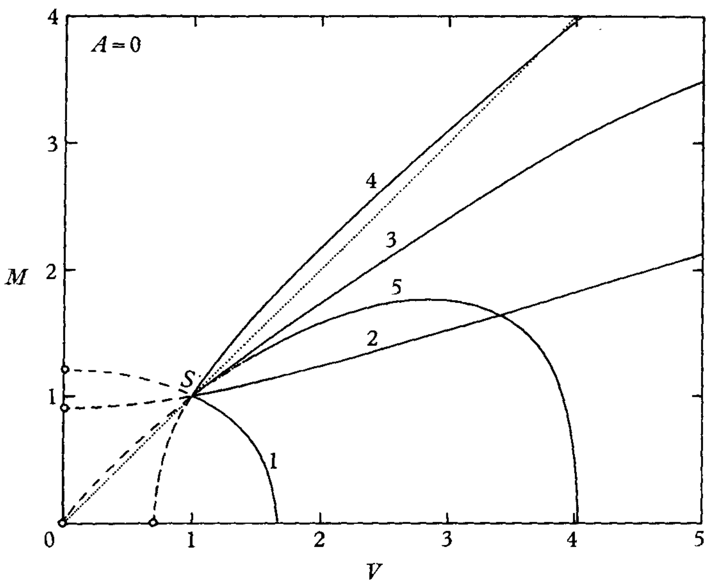

FIGURE 3. An M-V diagram for buoyant jets in a uniform still ambient fluid ($A = 0$). The curves shown represent: 1 flow from a source of heavy fluid ($B = -0{\cdot}462$) with negative buoyancy; 2 flow from a weak source of positive buoyancy ($B = 0{\cdot}185$); 3 the simple plume from a virtual source of pure buoyancy ($B = 0{\cdot}8$); 4 flow from a strong source of buoyancy ($B = 1{\cdot}6$), in which case the virtual source is of 'wake' type and the early stages of the actual flow are buoyancy dominated; 5 the simple plume in a stably stratified environment at rest, where again a maximum height is reached. Except in cases of negative buoyancy these flows are all ultimately jet-like.

satisfying $M = 1$ at $V = 1$ is required. An analytic solution might be found for this equation, but it is likely to be complicated in form and to need a good deal of numerical evaluation. Thus it may well be easier to solve the equation numerically—a simple task—and it is here that the use of an M–V diagram has some advantages since it permits a survey of the range of expected solutions with a minimum of labour. Particular cases which appear to have desired properties can subsequently be solved in detail. For example, the behaviour of jets with weak negative buoyancy might be investigated in this way to find the interaction between the retarding effect of the buoyancy and the accelerating effect of the entrained momentum on the jet. For present purposes it seems enough to give curves for the integrations: (i) $(A, B) = (0{\cdot}5, -1{\cdot}0)$, $(0{\cdot}5, -0{\cdot}0625)$ and $(0{\cdot}5, -0{\cdot}0025)$ with negative buoyancy, and (ii) $(A, B) = (0{\cdot}5, 0{\cdot}2)$ for positive buoyancy; these are shown on figure 4.

2.2. *Wakes*

This group includes all columnar flows with $u < u_0$ at $x = x_0$, the point which may be taken as the start of developed turbulent flow, and again includes buoyant columns that lie along the line of the buoyancy producing force field. The virtual

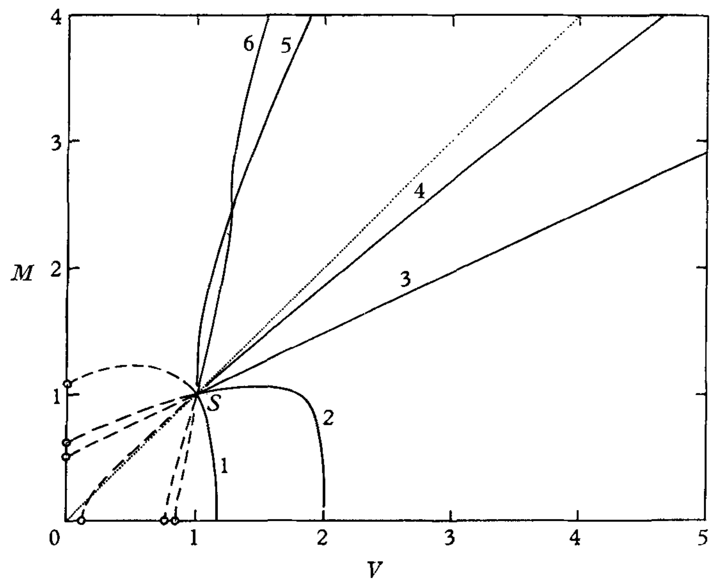

FIGURE 4. An M-V diagram for buoyant jets and wakes in a uniform stream. The curves shown represent: 1 a negatively buoyant jet ($A = 0{\cdot}5, B = -1{\cdot}0$) of heavy fluid with strong deceleration which rapidly reaches its greatest height ($M = 0$); 2 a weaker negatively buoyant jet ($A = 0{\cdot}5$, $B = -0{\cdot}0625$) which still suffers fairly rapid deceleration; 3 a buoyant jet with very weak negative buoyancy ($A = 0{\cdot}5$, $B = -0{\cdot}0025$) which is little affected; 4 a positively buoyant jet ($A = 0{\cdot}5$, $B = 0{\cdot}2$) where the discharge of buoyancy at the source S is sufficient to call for a 'wake-type' virtual source; 5 a forced wake ($A = 1{\cdot}2$, $B = 2$) with sufficiently strong positive buoyancy to cause transition from wake flow to jet flow close to the source S (shown by the infinite gradient there); 6 a more weakly buoyant forced wake ($A = 2$, $B = 1$) showing transition to jet flow at a rather greater 'distance'.

source for a wake has strength $(V, 0, B)$ in general, and since in this case $M \propto r^2u^2 \propto Vu$, it follows that when $x \to 0$, $u \to 0$ and $r \to \infty$, in such a way that there is a non-zero volume flux carrying zero momentum from the virtual source. This is, perhaps, a less commonly discussed type of virtual source, but as the flow predicted by the solution for $V < 1$ is purely hypothetical this is of no significance physically.

2.2(*a*) *Forced wakes* are column flows produced from a source emitting fluid at mean velocity u which is smaller than the free stream velocity u_0, so that $A > 1$; if there are no buoyancy effects $B = 0$, and equation (3) reduces to

$dM/dV = A$, subject to $M = 1$ at $V = 1$. These form a continuation of the solutions for jets in a uniform stream (case 2.1 *b*), with the same solution

$$M-1 = A(V-1).$$

Observable flow in the forced wake is again represented by the half-line of slope A through (1, 1) with $V > 1$, and the strength of the virtual source (of infinite area) is $\{(A-1)/A, 0\}$. Forced wake flows are possible for values of $A > 1$ lying in a range which is certainly bounded above, though it is not immediately clear what this upper bound will be; a typical example, calculated for $A = 1{\cdot}1$, is shown in figure 1. Equation (4) has solution satisfying $X = 0$ at $V = (A-1)/A$,

$$X = \tfrac{2}{3}A^{-1}(A-1)^{-1}[1+A(V-1)]^{\frac{3}{2}},$$

so that the effective source is at $X = \frac{2}{3}A^{-1}(A-1)^{-1}$. The full solution is

$$u = u_0\frac{X^{\frac{2}{3}}}{X^{\frac{2}{3}}+L^{\frac{2}{3}}}, \quad r = \frac{[(A-1)\,m_0]^{\frac{1}{2}}}{u_0}\,\frac{X^{\frac{2}{3}}+L^{\frac{2}{3}}}{L^{\frac{1}{3}}X^{\frac{1}{3}}},$$

where in this case $L = \frac{2}{3}(A-1)^{\frac{1}{2}}/A$; the asymptotic behaviour is

$$\frac{u-u_0}{u_0} \sim \frac{L^{\frac{2}{3}}}{X^{\frac{2}{3}}}\left\{1-\frac{L^{\frac{2}{3}}}{X^{\frac{2}{3}}}+\ldots\right\},$$

$$\frac{u_0 r}{[(A-1)\,m_0]^{\frac{1}{2}}} \sim \frac{X^{\frac{1}{3}}}{L^{\frac{1}{3}}}\left\{1+\frac{L^{\frac{2}{3}}}{X^{\frac{2}{3}}}\right\}.$$

2.2 (*b*). *The simple wake* is the limiting case of a forced wake and is produced when a bluff body modifies the flow of a steady uniform stream (or when a bluff body is moved at constant speed through a still environment, but here take a reference frame moving with the body). The modification to the uniform flow u_0 outside the wake will be disregarded. The simple wake is a special case of the forced wake for limiting A, and to find the value of A it is necessary to know the mean velocity in the wake at $x = x_0$. Between $x = x_0$ and the rear surface of the body there is an attached circulating flow, the nature of which depends strongly on the Reynolds number for flow past the body and on body shape (including the existence of a defined edge for separation), roughness, etc.; and except at small Reynolds numbers this circulating flow is bordered by a turbulent mixing layer which gradually thickens until it spreads completely across the wake at a point that is probably somewhat short of $x = x_0$ (there is some loss of fluid from the mixing layer to the circulating flow near this closing point). From this very rough description it may be expected that the appropriate value for A is likely to vary with Reynolds number (even at such relatively high values as 10^5) and with other factors; moreover, the task of finding A is made more difficult by the rather limited amount of experimental data which appears to be available for the attached flow. There is a temptation to take the virtual source actually at the rear of the body, but there is no possible justification for this even though u vanishes there. However, there are two ways in which A can be estimated. The first is to use direct measurements of wake velocity, for example using results for a circular flat plate normal to the stream and at Reynolds number

$1 \cdot 6 \times 10^5$ given by Fail, Lawford & Eyre (1957) of $u/u_0 = 0 \cdot 85$ approximately, at a distance 1·6 'attached bubble lengths' behind the plate (converting into terms of the top-hat profile); the corresponding value is $A = 1 \cdot 2$. The second method for estimating A is based on the drag of the body which is $\frac{1}{2} C_D \pi a^2 \rho u_0^2$ in terms of the drag coefficient C_D and the area πa^2 presented by the body to the oncoming stream, and is also $\rho \pi r^2 u(u_0 - u)$ in terms of the wake disturbance to the flow. Hence, to the present accuracy,

$$C_D = 2 \frac{r^2(A-1)}{a^2 A^2},$$

and A is given in terms of C_D, which is widely known, and the relative area of section of the wake downstream of the attached flow in terms of the area presented by the body, and this is as hard to find as the related information on wake velocities. The results of Fail and others are not easy to interpret for this purpose, but an estimate of the wake diameter at 1·6 attached bubble diameters indicates a value $A = 1 \cdot 3$ for the circular disk (with $C_D = 1 \cdot 12$); this agreement is probably much better than should be expected. Results for the sphere are more sensitive to Reynolds number, and the only really safe statement is that A is a good deal closer to unity, especially after transition to turbulence in the boundary layer. These results are of very limited use, but at least they show what information is needed if any particular problem is to be handled. A line is drawn in figure 1 for the simple wake with $A = 1 \cdot 25$, and this is probably close to the upper limiting line for axially symmetric simple wakes.

2.2 (*c*). *A buoyant forced wake* in a vertical stream of uniform density: the full equation (3) can again be solved numerically in a survey of solutions. Solution curves are shown in figure 4 for $A = 1 \cdot 2$, $B = 2$ and for $A = 2$, $B = 1$; the fluid in the wake is accelerated by entrainment of momentum and by buoyancy until it is moving with the ambient fluid (at which point dM/dV is infinite), and thereafter the behaviour changes over to that of a buoyant jet.

3. Application of the solution

How seriously should these solutions be taken? They add nothing to physical understanding of the flows described, and the formulation based on dimensional arguments has been reduced to such a state of simplicity that only the barest bones of the problem can have survived. Whenever more sophisticated methods of solution are available these will undoubtedly be superior; but the unfortunate fact is that more sophisticated methods of attack are often not available, especially for the more complicated problems which involve diffusion and convection of buoyant material along the column. In such cases the methods described will produce a solution which is correct at least in order of magnitude and usually much more closely, and the simplicity of the formulation ensures that the physical significance of the assumptions is exposed fully and is not likely to be overlooked in making an application.

Finally, a value or values must be chosen for the entrainment constant E. This question has already been discussed at some length by various authors (see Ellison & Turner (1959) and Morton (1959*b*)). Various values for E have been suggested for the case of jets and plumes in which the relative density variations are small; the differences are due partly to small variations in definition of the entrainment constant, and partly to the somewhat unequal quality of the available experimental results and to the difficulty in extracting accurately the required information from that which has been published. The results from jets suggest a value of about $E = 0{\cdot}116$ in the form used above; the less reliable results from plumes suggest a higher value although the difference is small if the unequal spread of momentum and heat is taken into account Values of E in cases of a moving environment are not at all easy to estimate, as there is a considerable scarcity of suitable results especially sufficiently far downstream to be of use, but it is hard to see why they should differ greatly from values for a simple jet. Moreover, E enters the various expressions mainly as a power with fractional exponent smaller than unity so that there is a tendency for errors in E to be reduced in importance. Thus while a good deal of further information on values of the entrainment constant is needed, the solutions given above are still of potential use in practical applications with E regarded as a known (or determinable) function of the column type.

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